Algebraic Geometry Lecture 7

1. Uniformisers

A few months ago in Dan's lecture we defined the order function that tells you whether a function has a zero or pole at a prime divisor (irreducible subvariety of codimension 1), and of what order. Given a prime divisor P of our variety and a rational function $f \in k(V)$ we defined

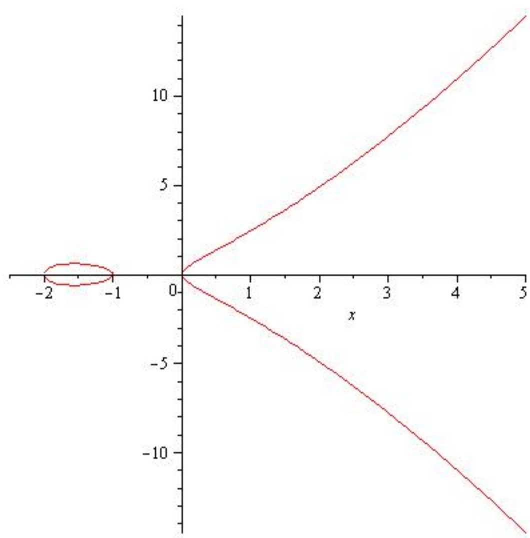
$$\operatorname{ord}_P(f) = \begin{cases} n > 0 & \text{if } f \text{ has a zero of order } n \text{ at } P, \\ 0 & \text{if } f \neq 0, \infty \text{ at } P, \\ -n < 0 & \text{if } f \text{ has a pole of order } n \text{ at } P. \end{cases}$$

Theorem 1. *The function* $\operatorname{ord}_P : k(V) \to \mathbb{Z} \cup \{\infty\}$ *is surjective.*

Def$^{\mathbf{n}}$. Given a prime divisor P of $V(\overline{k})$ we say a rational function $t \in \overline{k}(C)$ is a *uniformiser* of P if $\operatorname{ord}_P(t) = 1$.

Uniformisers exist by the theorem, since the ord_P map is surjective there must be a rational function of order 1. The theorem does not ensure uniqueness though and in general there are many choices we can take. Geometrically in the case of curves it is safe to think of uniformisers as being lines through the point P that are not tangential to the curve at that point.

Example Take the variety $V : y^2 - x^3 - 3x^2 - 2x = 0$ over $\mathbb{Q}$.



Consider the two points $P_1 = (1, \sqrt{6})$ and $P_2 = (-2, 0)$ in $V(\overline{\mathbb{Q}})$. For a uniformiser at P_1 we may take

$$t_1 : x - 1.$$

This has a simple zero at P_1 so it is a uniformiser there.

At P_2 we may be tempted to try

$$t_2 : x + 2.$$

This does have a zero at the point, so $\mathrm{ord}_{P_2}(t_2) \geqslant 1$, but we should note that

$$\begin{aligned} y^2 - x^3 - 3x^2 - 2x = 0 \\ \Rightarrow \quad x(x+1)(x+2) = y^2 \\ \Rightarrow \quad x + 2 = \frac{y^2}{x(x+1)}. \end{aligned}$$

So

$$\begin{aligned} \mathrm{ord}_{P_2}(x+2) &= \mathrm{ord}_{P_2}\left(\frac{y^2}{x(x+1)}\right) \\ &= \mathrm{ord}_{P_2}(y^2) - \mathrm{ord}_{P_2}(x) - \mathrm{ord}_{P_2}(x+1) \\ &= 2 - 0 - 0 \\ &= 2. \end{aligned}$$

So t_2 is not a uniformiser. But if we use $t_2' : y$ then we find that this rational function has a simple zero at P_2 so is a uniformiser there.

2. Divisors

Recall from Dan's lecture that, given a variety V over a field k, a divisor is just a formal sum of irreducible subvarieties of codimension 1 in $V(\overline{k})$ with coefficients in $\mathbb{Z}$. So in particular if V is a curve then divisors will be formal sums of points. The group of divisors is denoted $\mathrm{Div}(\overline{V})$.

Example If $V : y^2 - x^3 - 3x^2 - 2x = 0$ over $\mathbb{Q}$ then

$$(-2, 0) + 4(1, \sqrt{6}) - 19(17, 3\sqrt{646}) \in \mathrm{Div}(\overline{V}).$$

If we go through each subvariety in a divisor and replace the coordinates with their conjugates then we would expect to get a different divisor. If we in fact get the same divisor then the divisor is called k-rational. The subgroup of k-rational divisors is denoted $\mathrm{Div}(V)$.

Example Suppose that

$$(2, 0, 1) + 4(1, \sqrt{7}, 2 - 2\sqrt{7}) + 4(1, -\sqrt{7}, 2 + 2\sqrt{7}) \in \mathrm{Div}(\overline{V})$$

for some variety V. Then if we apply the embedding $\sqrt{7} \to -\sqrt{7}$ then we get

$$(2, 0, 1) + 4(1, -\sqrt{7}, 2 + 2\sqrt{7}) + 4(1, \sqrt{7}, 2 - 2\sqrt{7}).$$

But this is our original point, so this point is $\mathbb{Q}$-rational.

Recall: Using the order function we may associate a divisor to each rational function f by

$$\mathrm{div}(f) = \sum_{\substack{P \subset V(\overline{k}) \\ P \text{ prime divisor}}} \mathrm{ord}_P(f) P.$$

These divisors are called *principal divisors*.

3. Differentials

Motivation: In linear algebra you study vector spaces which are spaces of vectors. But vectors themselves aren't especially interesting, so you investigate maps and linear forms. In Algebraic Geometry we have varieties, which in themselves have limited appeal, so we investigate maps between them e.g. rational functions, and linear forms, which will be our differentials.

Def$^{\mathbf{n}}$. The space of differentials on a variety V is a vector space over the function field $k(V)$ with the 'vectors' being symbols dx for $x \in \overline{k}(V)$ subject to the relations

(1) $d(x+y) = dx + dy$
(2) $d(xy) = xdy + ydx$
(3) $da = 0$ for any $a \in k$.

The space of differentials is denoted Ω_V.

Theorem 2. *If V is an n-dimensional variety then Ω_V is an n-dimensional vector space over $k(V)$.*

Example What is $d\left(\frac{x}{y}\right)$? By property (ii) we get

$$d\left(\frac{x}{y}\right) = xd\left(\frac{1}{y}\right) + \frac{1}{y}dx.$$

So the question becomes, what is $d\left(\frac{1}{y}\right)$? By properties (iii) and (ii) we get

$$\begin{aligned} 0 &= d1 \\ &= d\left(\frac{y}{y}\right) \\ &= yd\left(\frac{1}{y}\right) + \frac{1}{y}dy, \end{aligned}$$

so that

$$d\left(\frac{1}{y}\right) = -\frac{1}{y^2}dy.$$

Thence

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}.$$

We now want to extend our notion of the order of a function at a point to cover differentials. To do this we need higher dimensional differentials. The differentials we've seen so far have been one-dimensional (though they may form a higher dimensional vector space). To get r-dimensional differentials we need to take the wedge product of them. This is just a formal product that allows us to "multiply" two differentials. The wedge product of two differentials dx and dy is denoted $dx \wedge dy$, and it satisfies the following rules for $dx, dy, dz \in \Omega_V$:

- $dx \wedge dy = -dy \wedge dx$
- $dx \wedge (dy + dz) = dx \wedge dy + dx \wedge dz$.

Using this product we can define the space of r-dimensional differentials, Ω_V^r as the space of products

$$dx_1 \wedge dx_2 \wedge \cdots \wedge dx_r$$

where dx_i are 1-dimensional differentials. If V has dimension n then Ω_V^r has dimension $\binom{n}{r}$ over $k(V)$.

If we have an n-dimensional differential ω on an n-dimensional variety then we can define its order and a corresponding divisor. It hinges on the fact that the dimension of Ω_V^n will be $\binom{n}{n} = 1$, so we can write ω in the form

$$\omega = f du_1 \wedge \cdots \wedge du_n$$

for some $f \in k(V)$. We then define

$$\text{ord}_P(\omega) = \text{ord}_P(f).$$

This is well defined despite the apparent ambiguity in choosing a basis vector. Using this notion we can go on to define the divisor of a differential as

$$\text{div}(\omega) = \sum_{\text{Prime divisors } P} \text{ord}_P(\omega) P.$$

These divisors are called the canonical divisors. It turns out that these divisors are invariant under conjugation, so they are in $\text{Div}(V)$. Moreover, the divisors of any two differentials differ by a principal divisor. Recall that we defined the divisor class group to be the group of k-rational divisors modulo the principal divisors. So all the canonical divisors lie in the same equivalence class of the divisor class group, known as the canonical class.

Def$^{\mathbf{n}}$. We say a differential $\omega \in \Omega_V$ is:

(1) regular (or holomorphic) if $\text{ord}_P(\omega) \geqslant 0$ for every prime divisor P;
(2) non-vanishing if $\text{ord}_P(\omega) \leqslant 0$ for every prime divisor P.

In the case of curves things simplify somewhat and any one-dimensional differential ω can be written in the form

$$\omega = f dt$$

for a rational function f and a uniformiser t.

Example Calculate $\text{ord}_P(dx)$ at $P = (-2, 0)$ for

$$C : y^2 - x^3 - 3x^2 - 2x = 0.$$

Recall we found that y was a uniformiser for C at P. So if we can write dx in the form $dx = f dy$ then $\text{ord}_P(dx) = \text{ord}_P(f)$. But:

$$x^3 + 3x^2 + 2x = y^2$$
$$(3x^2 + 6x + 2)dx = 2y dy$$
$$dx = \frac{2y}{3x^2 + 6x + 2} dy.$$

So

$$\begin{aligned}
\mathrm{ord}_P(dx) &= \mathrm{ord}_P\left(\frac{2y}{3x^2+6x+2}\right) \\
&= \mathrm{ord}_P(2) + \mathrm{ord}_P(y) - \mathrm{ord}_P(3x^2+6x+2) \\
&= 0 + 1 - 0 \\
&= 1.
\end{aligned}$$